Periodic Solutions for a Delayed Predator-Prey System with Harvesting Terms and Holling IV Functional Responses on Time Scales

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Abstract

In this paper, we investigate a delayed predator-prey system with Holling IV functional responses and harvesting terms on time scales. By using coincidence degree theory, we establish the existence of at least two positive periodic solutions for the above model.

Keywords: Periodic solution; Predator-prey system; Holling IV functional response; Coincidence degree theory; Time scales

1. Introduction

The predator-prey interactions play the most important role in the functioning of ecosystems. There are many different kinds of predator-prey models in the literature (Arditi and Ginzburg, 1989; Berryman, 1992; Ma, 1996; Li, 1993; Fan et al., 2003). According to different kinds of species on the foundation of experiments, many authors have concentrated on predator-prey systems with Holling IV functional response. For example, Chen and Zeng (2004) considered the following periodic predator-prey system with a type IV functional response:

\[
\begin{align*}
N'_1(t) &= N_1(t) \left[ b_1(t) - a_1(t)N_1(t) - \frac{c(t)N_2(t)N_2(t) - \sigma(t)}{(N_1^2(t)/n) + N_1(t) + a} \right], \\
N'_2(t) &= N_2(t) \left[ -b_2(t) + \frac{a_2(t)N_1(t)N_2(t) + \tau_2(t)}{(N_1^2(t)/n) + N_1(t) + a} \right],
\end{align*}
\]

(1.1)

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where $c$, $\sigma$, $a_j$, $b_j$ and $\tau_j$ $(j=1,2)$ are continuous $\omega$-periodic functions with $c(t) \geq 0$, $\sigma(t) \geq 0$, $a_j(t) \geq 0$ and $\tau_j(t) \geq 0$, $\int_0^\omega c(t)dt \geq 0$, $\int_0^\omega b_j(t)dt \geq 0$, $n$ and $a$ are positive constants. The growth functions $b_j$ are unnecessary to remain positive, since the environment fluctuates randomly. In bad conditions, $b_j$ may be negative. The author obtained sufficient conditions for the existence of multiple positive periodic solutions to (1.1) by applying the method of coincidence degree.

In order to unify continuous and discrete analysis, the theory of time scales was introduced by Hilger (1990). By choosing a time scale, general results can be applied to ordinary differential equations, and by choosing the time scale to the set of integers, they yield similar results for difference equations. So, He et al. (2009) considered the following periodic predator-prey system with a type IV functional response on time scale $T$:

$$
\begin{align*}
\dot{y}_1^\omega (t) &= b_1(t) - a_1(t) e^{\gamma(t)} - \frac{c(t) e^{\gamma(t)/n} e^{-\gamma(t)}}{e^{\gamma(t)/n} + e^{-\gamma(t)} + a}, \\
\dot{y}_2^\omega (t) &= -b_2(t) + a_2(t) e^{\gamma(t)} - \frac{c(t) e^{\gamma(t)/n} e^{-\gamma(t)}}{e^{\gamma(t)/n} + e^{-\gamma(t)} + a},
\end{align*}
$$

(1.2)

where for $i=1,2$, and $T$ is an $\omega$-periodic time scale, $y_i(t)$ and $\gamma_i(t)$ stand for the prey’s and predator’s density at time $t$, respectively; $c, \gamma, a_i, b_i, \tau_i \in C(\mathbb{R}_\omega)$ are $\omega$-periodic functions with $c(t) \geq 0$, $\gamma(t) \geq 0$, $a_i(t) \geq 0$, $\tau_i(t) \geq 0$, $c \geq 0$, and $b_i \geq 0$, $n$ and $a$ are positive constants. By means of coincidence degree theory, they establish the existence of at least two periodic solutions for the above model. Since the exploitation of biological resources and the harvest of population species are commonly practiced in fishery, forestry and wildlife management, the study of population dynamics with harvesting is an important subject in mathematical bioeconomics, which is related to the optimal management of renewable resources (Clark, 1990; Leung, 1995; Trowtman, 1996).

This motivates us to consider the following delayed predator-prey system with harvesting terms and Holling IV functional responses on time scales:

$$
\begin{align*}
\dot{y}_1^\omega (t) &= a(t) - b(t) e^{\gamma(t)/n} - \frac{c(t) e^{\gamma(t)/n} e^{-\gamma(t)}}{e^{\gamma(t)/n} + e^{-\gamma(t)} + m} - h_1(t) e^{-\gamma(t)}, \\
\dot{y}_2^\omega (t) &= d(t) + \frac{f(t) e^{\gamma(t)/n} e^{-\gamma(t)}}{e^{\gamma(t)/n} + e^{-\gamma(t)} + m} - h_2(t) e^{-\gamma(t)},
\end{align*}
$$

(1.3)

Where $T$ is an $\omega$-periodic time scale, $y_1(t)$ and $y_2(t)$ stand for the prey’s and predator’s density at time $t$, respectively; $a, b, c, d, f, \gamma, \tau_i, h_i \in C(T,\mathbb{R})$ $(i=1,2)$ are $\omega$-periodic functions with $c(t) \geq 0$, $\gamma(t) \geq 0$, $a_i(t) \geq 0$, $\tau_i(t) \geq 0$, $c \geq 0$, and $h_i \geq 0$, $m$ and $n$ are positive constants.
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\( b(t) \geq 0, \ d(t) \geq 0, \ f(t) \geq 0, \ \gamma(t) \geq 0, \ \tau_i(t) \geq 0, \ h_i(t) \geq 0 \ (i=1,2), \) for \( t \in T, \ t - \gamma(t), t - \tau_i(t) \in T \ (i=1,2), \) \( n \) and \( m \) are positive constants, \( h_1(t) \) and \( h_2(t) \) represent the harvesting rates of prey and predator, respectively.

The purpose of this paper is to obtain sufficient conditions for the existence of at least two periodic solutions to (1.3) by applying the method of coincidence degree. This is the first time that a delayed predator-prey system with a type IV functional response and harvesting terms has been studied by using this method.

2. Preliminaries

In this section, we shall recall some definitions and state some lemmas which will be used in the later section.

A time scale \( T \) is an arbitrary nonempty closed subset of the real numbers, and it inherits the topology from the real numbers with the standard topology. Throughout this paper, the time scale we considered is always assumed to be \( \omega \)-periodic (i.e., \( t \in T \) implies \( t \pm \omega \in T \)) and unbounded above and below. Set

\[
k = \min R^+ \cap T, \ I_\omega = [k, k + \omega] \cap T.
\]

**Definition 2.1:** [Bohner and Peterson(2001)] The forward and the backward jump operators \( \sigma, \rho : T \to T \), and the graininess \( \mu : T \to R^+ \) are defined, respectively, by

\[
\sigma(t) = \inf \{ s \in T : s > t \}, \ \rho(t) = \sup \{ s \in T : s < t \} \ \text{and} \ \mu(t) = \sigma(t) - t.
\]

We put \( \inf \Phi = \sup T \), and \( \sup \Phi = \inf T \), where \( \Phi \) denotes the empty set. A point \( t \) is said to be left-dense if \( t > \inf T \) and \( \rho(t) = t \), right-dense if \( t < \sup T \) and \( \sigma(t) = t \), left-scattered if \( \rho(t) < t \) and right-scattered if \( \sigma(t) > t \).

**Definition 2.2:** [Bohner and Peterson(2001)] Assume that \( f : T \to R \) and \( t \in T \), then \( f \) is called differential at \( t \in T \) if exists \( c \in R \) such that given any \( \varepsilon > 0 \), there is an open neighborhood \( U \) of \( t \) satisfying

\[
|f(\sigma(t)) - f(t) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,
\]

for all \( s \in U \). In this case, \( c \) is called the delta derivative of \( f \) at \( t \in T \), and is denoted by \( c = f^\Delta(t) \).

**Definition 2.3:** [Bohner and Peterson(2001)] We say that \( f \) is delta differentiable on \( T \) if \( f^\Delta(t) \) exists for all \( t \in T \). A function \( F : T \to R \) is called an antiderivative of \( f : T \to R \) provided that \( F^\Delta(t) = f(t) \) for all \( t \in T \). Then we define
\[ \int_{r}^{s} f(t) \Delta t = F(s) - F(r), \quad r, s \in T \]

**Lemma 2.1:** [Bohner and Peterson (2001)] If \( a, b \in T, \alpha, \beta \in R, \) and \( f, g \in C_{rd}(T) \), then

(a) \[ \int_{a}^{b} [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_{a}^{b} f(t) \Delta t + \beta \int_{a}^{b} g(t) \Delta t; \]

(b) if \( f(t) \geq 0 \), for all \( a \leq t < b \), then \( \int_{a}^{b} f(t) \Delta t \geq 0; \)

(c) if \( |f(t)| \leq g(t) \) on \( [a, b) = \{t \in T : a \leq t < b\} \), then \( \int_{a}^{b} f(t) \Delta t \leq \int_{a}^{b} g(t) \Delta t. \)

The following lemma cited from Agarwal et al. (2001); Bohner et al. (2006) which is useful for the proof of our main results of this paper.

**Lemma 2.2:** Let \( t_1, t_2 \in T \) and \( t \in T \). If \( g : T \to R \) is \( \omega \)-periodic, then

\[ g(t) \leq g(t_1) + \int_{t_1}^{t} |g^{\omega}(s)| \Delta s, \quad \text{and} \quad g(t) \geq g(t_2) - \int_{t_2}^{t} |g^{\omega}(s)| \Delta s. \]

For convenience, we denote

\[ \bar{g} = \frac{1}{\omega} \int_{t}^{t+\omega} g(s) \Delta s \]

where \( g \in C_{rd}(T) \) is an \( \omega \)-periodic function and introduce the following notation:

\[ u_{-} = \frac{\bar{a} - \sqrt{\bar{a}^{2} - 4 \bar{b} \bar{h} e^{-2\pi \omega}}}{2\bar{b}}, \quad l_{+} = \frac{\bar{a} + \sqrt{\bar{a}^{2} - 4 \bar{b} \bar{h} e^{-2\pi \omega}}}{2 \bar{b} e^{-2\pi \omega}}, \]

\[ p_{1} = \frac{\bar{h}_{2}}{f_{u_{-}}} + 2 \bar{f} \omega, \quad p_{2} = \frac{\bar{h}_{2}}{f_{u_{+}}} - 2 \bar{f} \omega, \quad p = \max \{|p_{1}|, |p_{2}|\}, \]

\[ l_{-} = \frac{(\bar{a} - \bar{e} e^{p}) + \sqrt{(\bar{a} - \bar{e} e^{p})^{2} - 4 \bar{b} \bar{h} e^{2\pi \omega}}}{2 \bar{b} e^{2\pi \omega}}, u_{+} = \frac{(\bar{a} - \bar{e} e^{p}) - \sqrt{(\bar{a} - \bar{e} e^{p})^{2} - 4 \bar{b} \bar{h} e^{2\pi \omega}}}{2 \bar{b}}. \]

Let \( X \) and \( Z \) be two Banach spaces, \( L : \text{Dom}L \subset X \to Z \) be a linear mapping and \( N : X \to Z \) be a continuous mapping. The mapping \( L \) will be called a Fredholm mapping of index zero if
\[ \dim \ker L = \dim \text{co} \dim \text{Im} L < +\infty, \quad \text{and} \quad \text{Im} L \text{ is closed in } Z. \]  
If \( L \) is a Fredholm mapping of index zero, then there exist continuous projectors \( P : X \to X \) and \( Q : Z \to Z \) such that \( \text{Im} P = \ker L \) and \( \text{Im} L = \ker Q = \text{Im}(I - Q) \). It follows that

\[
L \big|_{\text{Dom} L \cap \ker P} : (I - P)X \to \text{Im} L
\]

is invertible. We denote the inverse of the map \( L \big|_{\text{Dom} L \cap \ker P} \) by \( K_P \). If \( \Omega \) is an open bounded subset of \( X \), the mapping \( N \) will be called \( L \)-compact on \( \overline{\Omega} \) if \( \left( \text{Im} N \right)(\overline{\Omega}) \) is bounded and \( K_P(I - Q)N : \overline{\Omega} \to X \) is compact. Since \( \text{Im} Q \) is isomorphic to \( \ker L \), there exists an isomorphism \( J : \text{Im} Q \to \ker L \).

**Lemma 2.3:** [Gaines and Mawhin (1977)] Let \( L \) be a Fredholm mapping of index zero and let \( N \) be \( L \)-compact on \( \overline{\Omega} \). Suppose

1. \( \text{for each } \lambda \in (0,1), \) every solution \( x \) of \( Lx \neq \lambda Nx \) is such that \( x \in \partial \Omega \);
2. \( \text{Q}N \text{x} \neq 0, \forall x \in \ker L \cap \partial \Omega \);
3. \( \deg(JQNx, \Omega \cap \ker L, 0) \neq 0. \)

Then \( Lx = Nx \) has at least one solution in \( \text{Dom} \cap \partial \Omega \).

**3. Existence of periodic solutions**

In this section, our emphasis is focused on the existence of at least two periodic solutions for (1.3). Before formulate the main result, we first embed our problem into the frame of Lemma 2.3. Set

\[ X = Z = \{ y = (y_1, y_2)^T \in C(T, R^2) \mid y_1(t + \omega) = y_1(t), i = 1, 2, t \in T \}. \]

Then \( X, Z \) are Banach spaces endowed with the norm \( \|x\| = \sum_{i=1}^{2} \max_{t \in I}\|x_i(t)\| \).

Define

\[ N(y, \lambda) = (\Phi_1(t), \Phi_2(t))^T, \quad L\lambda(t) = (y_1^\lambda(t), y_2^\lambda(t)) \]

\[ Py = Qy = \left( \frac{1}{\omega} \int_{t_\omega} y_1(t) \Delta t, \frac{1}{\omega} \int_{t_\omega} y_2(t) \Delta t \right) \]

where \( y \in X \), and
\[
\Phi_1(t) = a(t) - b(t)e^{\gamma_1(t - \tau_1(t))} - \frac{\lambda c(t)e^{\gamma_2(t - \gamma(t))}}{e^{\gamma_1(t)}/n} + e^{\gamma_1(t)} + m - h_1(t)e^{-\gamma_1(t)},
\]
\[
\Phi_2(t) = -d(t) + \frac{f(t)e^{\gamma_1(t - \tau_2(t))}}{e^{\gamma_1(t - \tau_2(t))/n} + e^{\gamma_1(t - \tau_2(t)} + m - h_2(t)e^{-\gamma_2(t)}.
\]

Obviously,
\[
\text{Im } L = \{ y \in X : y = (k_1, k_2)^T \in \mathbb{R}^2 \}, \quad \text{Im } L = \{ y \in X : \int_{t_0}^{t} y_i(t) \Delta t = 0, i = 1, 2 \},
\]
then \( P, Q \) are continuous projectors such that
\[
\text{Im } P = \ker L, \quad \text{Im } L = \ker Q = \text{Im}(I - Q),
\]
the set \( \text{Im } L \) is closed in \( Z \), and \( \dim \ker L = 2 = \text{co dim } \text{Im } L \).
Hence, \( L \) is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to \( L \)) \( K_p : \text{Im } L \to \text{Dom } L \cap \ker P \) is given by
\[
K_p(Z) = \int_k^t z(s) \Delta s - \frac{1}{\omega} \int_{t_0}^{t} \int_k^s z(s) \Delta s \Delta t.
\]
Then
\[
QN(y, \lambda) = \begin{pmatrix}
\frac{1}{\omega} \int_{t_0}^{t} \Phi_1(s) \Delta s \\
\frac{1}{\omega} \int_{t_0}^{t} \Phi_2(s) \Delta s
\end{pmatrix}
\]
and
\[
K_p(I - Q)N(y, \lambda) = \begin{pmatrix}
\int_k^t \Phi_1(t) \Delta s - \frac{1}{\omega} \int_{t_0}^{t} \int_k^s \Phi_1(s) \Delta s \Delta t + \left( \frac{t - k - \frac{1}{\omega} \int_{t_0}^{t} (t - k) \Delta t} \Phi_1 \right) \\
\int_k^t \Phi_2(t) \Delta s - \frac{1}{\omega} \int_{t_0}^{t} \int_k^s \Phi_2(s) \Delta s \Delta t + \left( \frac{t - k - \frac{1}{\omega} \int_{t_0}^{t} (t - k) \Delta t} \Phi_2 \right)
\end{pmatrix}
\]

Obviously, \( QN \) and \( K_p(I - Q)N \) are continuous, and \( K_p(I - Q)N(\overline{\Omega}) \) is compact for any open bounded set \( \Omega \subset X \) by using the Arzelà-Ascoli theorem. Moreover, \( QN(\overline{\Omega}) \) is clearly bounded. Thus, \( N \) is \( L \)-compact on \( \overline{\Omega} \) with any open bounded set \( \Omega \subset X \).

**Theorem 3.1:** Suppose the following conditions hold:
\( (H_1) \quad \bar{a}, \bar{b}, \bar{c}, \bar{f} > 0; \)

\( (H_2) \quad \bar{a} > \bar{c}e^\rho + 2e^{\bar{a} \omega} \sqrt{\bar{b}h}; \)

\( (H_3) \quad \bar{f}u_\omega > \bar{d}(\bar{t}_n^2 / \bar{n}) + \bar{l}_n + \bar{m}. \)

Then system (1.3) has at least two \( \omega \)-periodic solutions.

Proof: Corresponding to the operator equation \( Ly = \lambda \mathcal{V}(y, \lambda), \lambda \in (0, 1) \), we have

\[
\begin{align*}
\begin{cases}
  y_1^\lambda(t) = \lambda \Phi_1(t) \\
  y_2^\lambda(t) = \lambda \Phi_2(t)
\end{cases} \tag{3.1}
\end{align*}
\]

Suppose that \( (y_1(t), y_2(t))^T \in T \) is a solution of system (3.1) for a certain \( \lambda \in (0, 1) \).

By integrating (3.1) over the set \( I_\omega \), we obtain

\[
\begin{align*}
\int_{I_\omega} \left[ a(t) - b(t) e^{y_1(t-\eta_1(t))} - \frac{\lambda c(t) e^{y_2(t-\eta_2(t))}}{(e^{y_1(t)/\bar{n}} + e^{y_2(t)/\bar{n}} + \bar{m})} - h_1(t) e^{-y_1(t)} \right] dt &= 0, \tag{3.2}
\end{align*}
\]

And

\[
\begin{align*}
\int_{I_\omega} \left[ -d(t) + \frac{f(t) e^{y_1(t-\eta_1(t))}}{(e^{y_1(t)/\bar{n}} + e^{y_2(t)/\bar{n}} + \bar{m})} - h_2(t) e^{-y_2(t)} \right] dt &= 0, \tag{3.3}
\end{align*}
\]

It follows from (3.1)-(3.3) that

\[
\int_{I_\omega} y_1^\lambda(t) \Delta t < \left\{ \int_{I_\omega} a(t) \Delta t + \int_{I_\omega} \left[ b(t) e^{y_1(t-\eta_1(t))} - \frac{\lambda c(t) e^{y_2(t-\eta_2(t))}}{(e^{y_1(t)/\bar{n}} + e^{y_2(t)/\bar{n}} + \bar{m})} - h_1(t) e^{-y_1(t)} \right] dt \right\} < 2\bar{a} \omega, \tag{3.4}
\]

And
\[ \int_{t_0}^{t_1} \left| y_i^\Delta(t) \right| \Delta t \]

\[ < \left\{ \int_{t_0}^{t_1} \left[ d(t) + h_2(t) e^{-\gamma_i(t)} \right] \Delta t + \int_{t_0}^{t_1} \left[ \frac{f(t) e^{\kappa(t - z_1(t))}}{h} + e^{\gamma_i(t - z_2(t))} + m \right] \Delta t \right\} \]

\[ < 2 \tilde{f} \omega. \]

Since \( (y_1, y_2)^\top \in X \), there exist \( \xi_i, \eta_i \in I_{\omega_i}, \ i = 1, 2 \), such that

\[ y_i(\xi_i) = \min_{t \in I_{\omega_i}} y_i(t), \quad y_i(\eta_i) = \max_{t \in I_{\omega_i}} y_i(t). \]

From (3.2) and (3.6) we obtain \( \tilde{\alpha} - \tilde{b} e^{\gamma_1(\xi_i)} - \tilde{h} e^{-\gamma_1(\eta_i)} > 0 \), which implies

\[ \tilde{b} e^{\gamma_1(\xi_i)} + \tilde{h} e^{-\gamma_1(\eta_i)} - \tilde{\alpha} < 0. \]

It follows from (3.4), (3.6) and Lemma 2.2 that

\[ y_1(t) \geq y_1(\eta_i) - \int_{t_0}^{t_1} \left| y_i^\Delta(t) \right| \Delta t > y_1(\eta_i) - 2 \tilde{\alpha} \omega. \]

In particular, we have

\[ y_1(\xi_i) > y_1(\eta_i) - 2 \tilde{\alpha} \omega, \]

By (3.7) and (3.8), we have that \( \bar{b} e^{2\pi \alpha - \gamma_1(\xi_i)} - \bar{a} e^{\gamma_1(\eta_i)} + \bar{h} i < 0 \). Because of \( (H_2) \), we obtain

\[ \ln \left( \bar{a} - \sqrt{\bar{a}^2 - 4 \bar{b} \bar{h} e^{-2\pi \omega}} \right) < y_i(\eta_i) < \ln \left( \bar{a} + \sqrt{\bar{a}^2 - 4 \bar{b} \bar{h} e^{-2\pi \omega}} \right) = \ln l_i. \]  

In view of (3.4), (3.6) and Lemma 2.2, we also have

\[ y_1(t) \leq y_1(\xi_i) + \int_{t_0}^{t_1} \left| y_i^\Delta(t) \right| \Delta t < y_1(\xi_i) + 2 \tilde{\alpha} \omega. \]

Particularly, we have

\[ y_1(\eta_i) < y_1(\xi_i) + 2 \tilde{\alpha} \omega, \]

thus (3.7) and (3.10) show that \( \bar{b} e^{2\gamma_1(\xi_i)} - \bar{a} e^{\gamma_1(\xi_i)} + \bar{h} e^{-2\pi \omega} < 0 \).

According to \( (H_2) \), we have
\[ \ln u_\pm = \ln \frac{\sqrt{\alpha^2 - 4\beta h_1 e^{-2\pi \omega}}}{2\beta} < y_1(\xi_1) < \ln \frac{\sqrt{\alpha^2 - 4\beta h_1 e^{-2\pi \omega}}}{2\beta}. \] (3.11)

From (3.2), (3.6), (3.9) and (3.11), we have

\[
\frac{\overline{f}_u_\pm}{(l^2/n) + l_\pm + m} < \frac{\overline{f}e^{\gamma(l)}(\xi)}{(e^{\gamma(n)/n}) + e^{\gamma(n)/m} + m} < \overline{d} + h_2 e^{-\gamma(\xi)},
\]

which implies \( y_2(\xi_2) < \ln \left[ \frac{\overline{h}_2}{\frac{\overline{f}_u_\pm}{(l^2/n) + l_\pm + m} - \overline{d}} \right]. \)

By (3.5), (3.6) and Lemma 2.2, we have

\[
y_2(t) \leq y_2(\xi_2) + \int_{t_\pm} y_2^\Delta(t) \Delta t < \ln \left[ \frac{\overline{h}_2}{\frac{\overline{f}_u_\pm}{(l^2/n) + l_\pm + m} - \overline{d}} \right] + 2\overline{f} \omega = p_1. \] (3.12)

From (3.3) and (3.6), we have

\[
\overline{h}_2 e^{-\gamma(\eta)} < \int_{t_\pm} f(t) e^{\gamma(t_\pm(t))} \Delta t < \overline{f},
\]

this implies \( y_2(\eta_2) > \ln(\overline{f}/\overline{h}_2) \). It follows from Lemma 2.1 that

\[
y_2(t) \geq y_2(\eta_2) - \int_{t_\pm} y_2^\Delta(t) \Delta t > \ln(\overline{f}/\overline{h}) - 2\overline{f} \omega = p_2. \] (3.13)

Then from (3.12) and (3.13), we have

\[
\max_{t_\pm} y_2(t) \leq \max \left\{ p_1, p_2 \right\} = p. \] (3.14)

On the other hand, from (3.2), (3.4), (3.6) and (3.14), we also have

\[
\overline{a} < \overline{b} e^{\gamma(n)} + \overline{c} e^{\nu} + \overline{h}_2 e^{-\gamma(\xi)}, \text{ which implies } \overline{b} e^{\gamma(n)} + \overline{h}_2 e^{-\gamma(\xi)} - (\overline{a} - \overline{c} e^{\nu}) > 0. \] (3.15)
In view of (3.4), (3.6) and Lemma 2.2, we have
\[ y_1(t) \geq y_1(\eta) - \int_{\eta}^{t} y_1^\lambda(t) \Delta t > y_1(\eta) - 2\bar{a} \omega. \]
In particular, we have
\[ y_1(\xi) > y_1(\eta) - 2\bar{a} \omega. \tag{3.16} \]
Thus (3.15) and (3.16) show that \( \overline{b}e^{2y_1(\eta)} - (\bar{a} - \bar{c}e^p) e^{y_1(\eta)} + \bar{h}_1 e^{2\bar{a} \omega} > 0 \). Because of \( (H_2) \), we have
\[ y_1(\eta) < \left( \frac{a - ce^p}{2b} \right) + \sqrt{\left(\frac{a - ce^p}{2b}\right)^2 - \frac{4b \bar{h}_1 e^{2\bar{a} \omega}}{2b}} = \ln u, \tag{3.17} \]
Or
\[ y_1(\eta) > \ln \left( \frac{a - ce^p}{2b} \right) + \sqrt{\left(\frac{a - ce^p}{2b}\right)^2 - \frac{4b \bar{h}_1 e^{2\bar{a} \omega}}{2b}}. \]
By (3.4), (3.6) and Lemma 2.2, we also have
\[ y_1(t) \leq y_1(\xi) + \int_{\xi}^{t} y_1^\lambda(t) \Delta t \leq y_1(\xi) + 2\bar{a} \omega. \]
Particularly, we have
\[ y_1(\eta) < y_1(\xi) + 2\bar{a} \omega. \tag{3.18} \]
Thus (3.15) and (3.18) show that
\[ \overline{b}e^{2\bar{a} \omega} e^{2y_1(\xi)} - (\bar{a} - \bar{c}e^p) e^{y_1(\xi)} + \bar{h}_1 > 0. \]
According to \( (H1) \), we have
\[ y_1(\xi) < \ln \left( \frac{a - ce^p}{2b} \right) - \sqrt{\left(\frac{a - ce^p}{2b}\right)^2 - \frac{4b \bar{h}_1 e^{2\bar{a} \omega}}{2b}} \]
or
\[ y_1(\xi) > \ln \left( \frac{a - ce^p}{2b} \right) + \sqrt{\left(\frac{a - ce^p}{2b}\right)^2 - \frac{4b \bar{h}_1 e^{2\bar{a} \omega}}{2b}} = \ln \bar{l}. \tag{3.19} \]
We claim that $u_- < u_+, \ l_- < l_+$ and $u_+ < l_+$. Construct a function

$$g(x) = x - \sqrt{x^2 - a}, \ a > 0 \text{ and } x > \sqrt{a}.$$ 

Then

$$g'(x) = \frac{-g(x)}{\sqrt{x^2 - a}} < 0,$$

which implies that

$$\frac{-a - \sqrt{a^2 - 4bh_1e^{2\alpha_0}}}{2b} < \frac{(a - ce^p) - \sqrt{(a - ce^p)^2 - 4bh_1e^{2\alpha_0}}}{2b}, \quad (3.20)$$

Since

$$\frac{-a - \sqrt{a^2 - 4bh_1e^{2\alpha_0}}}{2b} > \frac{-a - \sqrt{a^2 - 4bh_1e^{-2\alpha_0}}}{2b}. \quad (3.21)$$

By (3.20) and (3.21), also notice (3.11) and (3.17), we know

$$u_- = \frac{-a - \sqrt{a^2 - 4bh_1e^{2\alpha_0}}}{2b} < \frac{-a - \sqrt{a^2 - 4bh_1e^{-2\alpha_0}}}{2b} = u_+.$$ 

In addition, constructing two functions

$$h(x) = \frac{a + \sqrt{a^2 - bx}}{x}, \ 0 < x < \frac{a^2}{b}, \ a > 0, \ b > 0$$

And

$$s(x) = x + \sqrt{x^2 - a}, \ a > 0 \text{ and } x > \sqrt{a},$$

it is easy to know that

$$h'(x) < 0 \text{ and } s'(x) > 0,$$

Thus
\[ l_+ = \frac{\bar{a} - ce^p}{e^2} \pm \sqrt{\frac{(\bar{a} - ce^p)^2}{e^2} - 4\bar{b}h_1e^{-2\omega}} \]

\[ < \frac{\bar{a} - ce^p}{e^2} + \sqrt{\frac{(\bar{a} - ce^p)^2}{e^2} - 4\bar{b}h_1e^{-2\omega}} \]

Also through some simple calculation, we can show that when \((H2)\) holds true, then \(u_+ < l_+\) also holds. From (3.9), (3.11), (3.17) and (3.19), we have for any \(t \in l_\alpha\),

\[ \ln u_- < y_1(t) < \ln u_+ \text{ or } \ln l_- < y_1(t) < \ln l_+ \quad (3.22) \]

Clearly, \(\ln u_-\), \(\ln u_+\), \(\ln l_-\), \(\ln l_+\) and \(P\) are independent of the choice of \(\lambda \in (0,1)\).

Now, consider \(QN(y,0)\) with \(y = (y_1, y_2)^T \in R^2\). Note that

\[
QN(y_1, y_1'; 0) = \begin{pmatrix}
\bar{a} - be^{y_1} - h_1e^{-y_1} \\
-d + \frac{fe^{y_1}}{(e^{2y_1}/n) + e^{y_1} + m} - h_2e^{-y_2}
\end{pmatrix}
\]

Since \((H_2), (H_3)\), we can show that \(QN(y_1, y_2'; 0) = 0\) has two distinct solutions:

\[ (y_1^1, y_2^1) = (\ln \nu_-, \ln \frac{h_2}{g(\nu_-) - d}), \ (y_1^2, y_2^2) = (\ln \nu_+, \ln \frac{h_2}{g(\nu_+) - d}), \]

Where \(\nu_+ = \frac{a \pm \sqrt{a^2 - 4bh_1}}{2b}, \ g(\nu_-) = \frac{-f\nu_-}{\nu_-^2/n + \nu_- + m}, \ g(\nu_+) = \frac{-f\nu_+}{\nu_+^2/n + \nu_+ + m} \).

Obviously,

\[ \ln u_- < \ln \nu_- < \ln u_+ < \ln l_- < \ln \nu_+ < \ln l_+. \]

Chose \(M > 0\) such that
\[
M > \max \left\{ \left| \ln \frac{\tilde{h}_2}{g(\nu_1) - d} \right|, \left| \ln \frac{\tilde{h}_2}{g(\nu_2) - d} \right| \right\}. \tag{3.23}
\]

And set
\[
\Omega_1 = \{ y = (y_1, y_2)^T \in X : y_1 \in (\ln u_-, \ln u_+), \max_{i = 1, 2} |y_2| < p + M \}
\]

And
\[
\Omega_2 = \{ y = (y_1, y_2)^T \in X : y_1 \in (\ln l_-, \ln l_+), \max_{i = 1, 2} |y_2| < p + M \},
\]

it is easy to see that \((y_1^1, y_2^1) \in \Omega_1, \ (y_1^2, y_2^2) \in \Omega_2 \) and \(\Omega_i (i = 1, 2)\) are open bounded subset of \(X\). With the help of (3.9)-(3.11), (3.13), (3.14) and (3.17)-(3.23), it is not difficult to show that \(\Omega_1 \cap \Omega_2 = \emptyset\) and \(\Omega_i\) verify the requirements (1) of Lemma 2.3 for \((i = 1, 2)\). Moreover, when \(y \in \partial \Omega_i \cap \ker L, i = 1, 2, \ QN(y, 0) \neq (0, 0)^T\), so condition (2) of Lemma 2.3 holds.

Finally, we will verify that condition (3) of Lemma 2.3 holds. By taking \(J = I\) since \(\ker L = \text{Im} Q\), a direct computation gives for \(i = 1, 2\),

\[
\deg (JQN(y, 0), \Omega_i \cap \ker L(0, 0)^T) = \text{sign} \left[ \begin{array}{ccc}
-\tilde{b}e^{y_1} + \tilde{h}_1 & 0 \\
-\tilde{f}e^{y_1}(\frac{\tilde{y}_1}{\tilde{e}} - m) & \tilde{h}_2 \\
\frac{e^{y_1} -m}{(e^{y_1} + m)} & e^{y_2}
\end{array} \right]
\]

\[
= \text{sign} \left[ \begin{array}{c}
-\tilde{b}e^{y_1} + \tilde{h}_1 \\
\tilde{h}_2
\end{array} \right] e^{y_2}
\]

\[
= \text{sign} \left[ -\tilde{b}e^{y_1} + \tilde{h}_1 \right] e^{y_2},
\]

together with the fact
\[
\begin{align*}
\tilde{a} - \tilde{b}e^{y_1} - \tilde{h}_1e^{-y_1} &= 0, \\
-\tilde{d} + \frac{\tilde{f}e^{y_1}}{(e^{y_1}/n) + e^{y_1} + m} - \tilde{h}_2e^{-y_2} &= 0,
\end{align*}
\]

Then
\[
\deg (JQN(y, 0), \Omega_i \cap \ker L(0, 0)^T) = \text{sign}(\tilde{a} - 2\tilde{b}e^{y_1}), (i = 1, 2).
\]
Thus
\[ \text{deg}(JQN(y, 0), \Omega_1 \cap \ker L(0, 0)^T) = \text{sign}(a - 2b) = 1, \]
\[ \text{deg}(JQN(y, 0), \Omega_2 \cap \ker L(0, 0)^T) = \text{sign}(a - 2b) = -1. \]

So far, we have proved that \( \Omega(i = 1, 2) \) satisfies (1)-(3) of Lemma 2.3. Hence, the system (1.3) has at least two positive \( \omega \)-periodic solutions. The proof of Theorem 3.1 is complete.

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