Two-sided Jump-diffusion Model with Correlated Jumps under the Barrier Dividend Strategy

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Abstract

This paper studies the perturbed classical risk model with two dependent classes of jumps and a constant dividend barrier. We use a double exponential jump-diffusion process to model the fluctuation of the total claim amount, the premium income and the surplus investment return, and the claim process follows a compound Poisson process which is dependent of the double exponential jump-diffusion process. We can prove a type of expected discounted penalty function satisfies the smooth pasting property at the reflecting barrier under the two-sided jump-diffusion model. We give the explicit expression for the joint Laplace transform of the ruin time and the deficit at ruin time.

Keywords: Reflected jump-diffusion process; Barrier strategy; Ruin time; Smooth pasting; Rational Laplace transform

1. Introduction

In recent years, various dividend payment strategies for the classical collective risk model have been studied in great detail. Under a barrier strategy, when the surplus of an insurance company reaches a barrier level, premium income no longer goes into the surplus but is paid out as dividends to shareholders. Such a dividend-payment strategy was first discussed in De Finetti (1957) for a Bernoulli model. In the past decade, there has been an enormous amount of interest in studying the generalized Cramer-Lundberg risk model without and with diffusion under the barrier dividend
strategy, see, for example, Zhou (2005), Lin et al (2003), Avram et al. (2007), Loeffen (2008), and Loeffen (2009). All of the above mentioned work is based on one-sided models.

Recently, two-sided jump-diffusion models attract more and more attention, see, for example, Yin and Yuen (2011), Yuen and Yin (2011), and Bo et al (2012). Under a two-sided jump-diffusion model, the upward jumps can be interpreted as random returns of an insurance company, while the downward jumps are interpreted as random losses or claims.

In recent years, studies of an insurance risk model have in general been focusing on analyzing the Gerber-Shiu expected discounted penalty function, which is first introduced by Gerber and Shiu (1998) and Gerber and Laudry (1998) to analyze the quantities, such as, the time of ruin, the surplus immediately before ruin and the deficit at ruin, in a unified manner. It has proven to be a powerful analytical tool. The Gerber-Shiu function has been fully studied in the compound Poisson risk models, and the one-sided jump-diffusion models with and without dividend. But it is in general not easy to give a closed-form formula for the Gerber-Shiu function in a two-sided jump-diffusion model with the dividend strategy.

The double exponential jump-diffusion process, firstly proposed by Kou (2002), Kou and Wang (2004), is used to model the returns of the stocks. Under a constant dividend barrier strategy, Bo et al (2012) derive some explicit expressions for the Laplace transform of the ruin time, the distribution of the deficit at ruin, and the total expected discounted dividends when the surplus of a company follows a double exponential jump-diffusion process. In this paper, motivated by Bo et al (2012), Kou (2002), and Kou and Wang (2004), we assume the surplus process of an insurance company is expressed as a sum of a double exponential jump-diffusion process and a compound process with only negative jumps; the double exponential jump-diffusion process describes the fluctuation of the total claim amount, the premium income and the surplus investment return and the compound process with only negative jumps models the total claims. We aim at obtaining the expression for the Gerber-Shiu function under a constant barrier dividend strategy.

In order to derive the explicit formula for the Gerber-Shiu function, we assume the distribution of the claim sizes have a rational Laplace transform. Under this assumption, we can give the closed-form formula for the Gerber-Shiu function. In fact, the rational family distributions are rich enough to approximate many other distributions, including any discrete distribution, the normal distribution, and various heavy-tailed distributions such as Gamma, Weibull and Pareto distributions. Therefore, our model is even more flexible and can be used to approximate any model in which the downward jump sizes have an arbitrary distribution and the upward jump sizes have an exponential distribution.

The paper is organized as follows: Section 2 describes the model we consider in this paper, and establishes a key quantity between the joint Laplace transform of ruin time and the deficit, and the joint Laplace transform of first passage time and the overshoot. Section 3 presents some preliminary results. The explicit formula for the joint Laplace transform of ruin time and the deficit is derived in Section 4. Section 5 gives some numerical results and concludes.
2. The Model

Given a filtered complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\), all random variables and stochastic processes of this paper are assumed to be defined on it. Consider a surplus process of a firm given by:

\[
U_t = u + ct + \sigma W_t + \sum_{i=1}^{N_1(t)} Z_i - \sum_{i=1}^{N_2(t)} Y_i \quad u + ct + \sigma W_t + S_t
\]  

(2.1)

where \(u > 0\) is the initial firm value, \(c > 0\) is the rate of premium; \(\{W_t, t \geq 0\}\) is a Brownian motion; \(\{N_1(t), t \geq 0\}\) is a Poisson process, \(\{Z_i, i \geq 1\}\) is a sequence of independent and identically distributed random variables, and independent of other variables, with the common density given by

\[
f_1(z) = p_0 \alpha e^{-\alpha z} 1_{z > 0} + q_0 \beta e^{-\beta z} 1_{z < 0},
\]

(2.2)

where \(0 < p_0, q_0 < 1, p_0 + q_0 = 1, \alpha > 0, \beta > 0\). In fact, the process \(\sigma W_t + \sum_{i=1}^{N_1(t)} Z_i\) is a double exponential jump-diffusion process, which is used to model the returns of the stocks, as in Kou (2002), Kou and Wang (2004). In this paper, we use the double exponential jump-diffusion model to describe the fluctuation of the total claim amount, the premium income and the surplus investment return. \(\{N_2(t), t \geq 0\}\) is another Poisson process counting the claim number in the interval \((0, t]\), \(\{Y_i, i \geq 1\}\), also independent of other variables, is a sequence of independent and identically distributed positive random variables representing the successive individual claim amounts and having the common density function \(f_2(y)\) with rational Laplace transform given by

\[
\hat{f}_2(y) = \int_0^\infty e^{-sy} f_2(y) dy = \frac{v(s)}{\prod_{i=1}^m (v_i + s)^{n_i}}
\]

(2.3)

where \(m, n_i\) are positive integers with \(n_1 + n_2 + \cdots + n_m = n\), \(v_i \neq v_j > 0, i \neq j\) and \(v_i \neq \alpha, \beta\) for \(i = 1, \cdots, m\). \(v(s)\) is a polynomial function of degree \(n-1\) or less satisfying

\[
v(0) = \prod_{i=1}^m v_i^{n_i}.
\]

Furthermore, we assume that the following net profit condition holds, i.e.,

\[
ct + E[S_t] > 0.
\]

By partial fraction, (2.3) can be rewritten as follows
\begin{align*}
\hat{f}_2(y) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\beta_{ij} y^j}{(v_i + s)^j} \\
(2.4)
\end{align*}

where \( \beta_{ij} = \frac{1}{v_i^j (n_j - j)!} \left\{ \prod_{k=1, k \neq i}^{m} \frac{v(s)}{(v_k + s)^{n_k}} \right\} |_{s=-v_i} \).

Inverting the Laplace transform (2.4), the density \( f_2(y) \) can be expressed as a combination of Erlangs

\begin{align*}
f_2(y) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \beta_{ij} y^j \frac{1}{(j-1)!} e^{-v_i y}, y > 0
(2.5)
\end{align*}

Obviously, the model (2.1) is a generalization of the classical risk process perturbed by diffusion introduced by Gerber (1970) in insurance mathematics. Chi (2010) considers a similar model, but he assumes the jump processes \( N_1(t) \) and \( N_2(t) \) are independent. The double exponential jump-diffusion process incorporates the fluctuation of the total claim amount, so \( N_1(t) \) and \( N_2(t) \) may jump together. In this paper we assume

\begin{align*}
N_1(t) = N_{11}(t) + N_{12}(t), N_2(t) = N_{22}(t) + N_{12}(t)
(2.6)
\end{align*}

where \( N_{11}(t), N_{12}(t) \) and \( N_{22}(t) \) are three mutually independent Poisson processes with parameters \( \lambda_1, \lambda_{12} \) and \( \lambda_2 \), respectively. The assumption (2.6) is the so-called “common shock structure” discussed in Cossette and Marceau (2000). We will consider the ruin problem for the generalized classical risk process perturbed by diffusion under a constant barrier dividend.

Bo et al. (2012) give the explicit expressions for the Laplace transform of the ruin time and the distribution of the deficit at ruin under the double exponential jump-diffusion model with a barrier dividend strategy. The model (2.1) is more general than the double exponential jump-diffusion model. Motivated by Bo et al. (2012), we will consider the expected discounted penalty function under a barrier dividend strategy.

Let \( D_t \) be the aggregate dividends paid from 0 to \( t \), then, under the barrier dividend strategy, the aggregate dividends \( D_t \) can be expressed as,

\begin{align*}
D_t = \sup_{0 \leq s \leq t} (U_s - b)^+
(2.7)
\end{align*}

where \( b > U_{0+} \) is the constant dividend barrier. Since the risk model (2.1) is a cadlag process, it is separable and hence \( D_t \) is well-defined. Let
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\[ U^b_t = U_t - D_t \leq b \]  
\[ (2.8) \]

be the surplus process regulated by the dividend payment \( D_t \). Furthermore, the dividend process \( D_t \) can be rewritten as

\[ D_t = \int_{[0,t]} 1_{\{U^b_s = b\}} dD_s. \]  
\[ (2.9) \]

Define the ruin time as \( \hat{\tau}_u = \inf\{t : U^b_t \leq 0\} \), with \( \hat{\tau}_u = +\infty \), if \( U^b_t > 0 \) for all \( t \). Define the expected discounted penalty function, introduced by Gerber and Landry (1998), as

\[ \Psi(u) = E[e^{-\delta\hat{\tau}} \eta(U^b_{\hat{\tau}}) | U^b_0 = u], \]  
\[ (2.10) \]

where \( \delta > 0 \) is interpreted as the force of interest or the variable of a Laplace transform and \( \eta(.) \) is a non-negative function defined on \([0, +\infty)\). We remark that the penalty function \( \eta(.) \) is not necessarily continuous at 0. When the function \( \eta(.) \) is chosen as \( \eta(x) = e^{\alpha x}, \eta < 0 \), the expected discounted penalty function

\[ \Psi(u) = E[e^{-\delta\hat{\tau}} \eta(U^b_{\hat{\tau}}) | U^b_0 = u] \]

is also regarded as the joint Laplace transform of ruin time and the deficit. In this paper, we aim at deriving the explicit expression for the joint Laplace transform.

To this end, we introduce the following jump-diffusion process

\[ X_t = U_t - b = u - b + ct + \sigma W_t + S_t. \]  
\[ (2.11) \]

Define \( Y_t = M_t - X_t \) as \( X \) reflected at its running supremum \( M \), where \( M_t = \sup_{0 \leq s \leq t} X_s \vee 0 \). It is easy to check

\[ U^b_t = b - Y_t. \]  
\[ (2.12) \]

Define the entrance time of \( Y \) into \([b, +\infty)\)

\[ \tau_b = \inf\{t : Y_t \geq b\}, \]  
\[ (2.13) \]

with \( \inf \phi = \infty \).
Then from the definitions of $\hat{u}_\tau$ and $\tilde{u}_\tau$, we can conclude
\[
\Psi(u) = \Phi(x),
\] (2.14)
where
\[
\Phi(x) = E[e^{-\theta_0 Y_x} | Y_0 = x], x = b - u.
\]
For simplicity, denote by the conditional expectation operator $E_y[.] = E[.| Y_0 = x]$, with $E[.] = E[.| Y_0 = 0]$. Next we will firstly derive the expression for $\Phi(x)$.

3. Preliminary Results

In this section, we first analyze the process $\{S_t, t \geq 0\}$ in the model (2.1).

\textbf{Lemma 3.1} Assume that $N_1(t) = N_{11}(t) + N_{12}(t)$, $N_2(t) = N_{22}(t) + N_{12}(t)$, where $N_{11}(t)$, $N_{12}(t)$ and $N_{22}(t)$ are three mutually independent Poisson processes with parameters $\lambda_1$, $\lambda_{12}$ and $\lambda_2$, respectively. Then the process $S_t = \sum_{i=1}^{N_1(t)} Z_i - \sum_{i=1}^{N_2(t)} Y_i = \sum_{i=1}^{N(t)} X_i$ is a compound process with parameter and common density given by $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$ and
\[
f(x) = \begin{cases} 
p_1 e^{-\alpha x}, & x \geq 0, \\
q_1 e^{-\beta x} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} q_{ij} \frac{(-x)^{j-1} v_j e^{v_j x}}{(j-1)!}, & x < 0,
\end{cases}
\] (3.1)

With
\[
p_1 = \left(\frac{\lambda_1}{\lambda} + \frac{\lambda_{12} p}{\lambda}\right) p_0, p_0 = \frac{v(\alpha)}{\prod_{i=1}^{m} (v_i + \alpha)^{n_i}}, 
q_1 = \left(\frac{\lambda_1}{\lambda} + \frac{\lambda_{12} q}{\lambda}\right) q_0, q_0 = \frac{v(-\beta)}{\prod_{i=1}^{m} (v_i - \beta)^{n_i}},
\]
\[
q_{ij} = \frac{\lambda_2 f_x}{\lambda} + \frac{\lambda_{12} (u_x + v_y)}{2\lambda},u_y = \frac{1}{v_x(n_x - j)!} \left. \frac{d^{n_x-j}}{ds^{n_x-j}} \left( \prod_{k=1,k \neq i}^{m} \frac{-\beta v(s)}{(\beta + s)(v_k + s)^{n_k}} \right) \right|_{s=-v_x},
\]
\[
v_y = \frac{1}{v_y(n_y - j)!} \left. \frac{d^{n_y-j}}{ds^{n_y-j}} \left( \prod_{k=1,k \neq i}^{m} \frac{\alpha v(s)}{(\alpha - s)(v_k + s)^{n_k}} \right) \right|_{s=-v_y}.
\]

\textbf{Proof.} The moment generating of $S_t$ is given by
By partial fraction, we have

\[
\frac{\beta v(s)}{(\beta + s) \prod_{i=1}^{m} (v_i + s)^n} = q \frac{\beta}{\beta + s} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} u_{ij} \frac{v_i^j}{(v_i + s)^j},
\]

And

\[
\frac{\alpha v(s)}{(\alpha - s) \prod_{i=1}^{m} (v_i + s)^n} = p \frac{\alpha}{\alpha - s} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} v_i^j \frac{v_i^j}{(v_i + s)^j}.
\]

Substituting the above formulas and (2.5) into (3.2) yields Lemma 3.1.

Define the Laplace exponent of the jump-diffusion model \( U_t - u \) as

\[
g(s) = \ln E[e^{s(U_t - u)}] = \frac{1}{t} \sigma^2 + cs - \lambda + \lambda q_0 + \beta \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{q_i \beta}{(s + v_i)^j} + \lambda p_i \frac{\alpha}{\alpha - s}.
\]

In risk theory, the equation

\[
g(s) = \delta, \delta > 0,
\]

is called the generalized Lundberg equation. The following result makes some analysis on the roots of Eq. (3.3).

**Lemma 3.2** For \( \delta > 0 \), Eq. (3.3) has exactly \( n + 2 \) roots, \( r_1, r_2, \ldots, r_{n+2} \), on the left-half complex plane and two roots \( r_{n+3}, r_{n+4} \) with \( 0 < r_{n+3} < \alpha < r_{n+4} \) on the right-half complex plane.
Proof. From Lemma 1 in Zhang et al. (2010), we can conclude the equation (3.3) has exactly two roots on the right-half complex plane. Since (3.3) has exactly $n + 4$ roots, it has $n + 2$ roots on the left-half complex plane. And note $g(\alpha^-) = +\infty$, $g(\alpha^+) = -\infty$, $g(0) = 0$, $g(+\infty) = +\infty$, so the two roots on the right-half complex plane of Eq. (3.3) satisfy $0 < r_{n+3} < \alpha < r_{n+4}$.

Remark. The roots of Eq. (3.3) play an important role in the rest of this paper. For simplicity, we only consider the case when they are all distinct since the analysis of the other case is more tedious.

In order to derive the explicit express for $\Phi(x)$, we will prove the smooth pasting property $\Phi'(0+) = 0$ holds under the model (2.1). Bo et al. (2012) prove the smooth pasting condition $\Phi'(0+) = 0$ holds under the double exponential jump-diffusion model. Now we give the smooth pasting property at the reflecting barrier under the general model considered in our paper. To this end, we rewrite $X$ defined in (2.11) as

$$X_i = X_i^{(+)} + X_i^{(-)}.$$  \hspace{1cm} (3.4)

where $X_i^{(-)}$ is a pure jump Levy process which can jump only downward, $X_i^{(+)} = \sum_{i=1}^{N_i^{(-)}} X_i^{(-)} I_{[X_i < 0]}$ is a spectrally positive Levy process. Then the process $-X_i^{(+)}$, which is the dual process of $X_i^{(+)}$, is a spectrally negative Levy process. Denote by $Y^{(+)} = M^{(+)} - X^{(+)}$ the spectrally positive Levy process $X^{(+)}$ reflected at its running supremum

$$M^{(+)} = \sup_{0 \leq s \leq t} X_s^{(+)} \vee 0.$$  

Define the entrance time of the reflected processes $Y^{(+)}$ into $(y, +\infty)$ as

$$\tau^{(+)}_y = \inf\{t \geq 0 : Y_t^{(+)} > y\} = \inf\{t \geq 0 : -X_t^{(+)} - \inf_{0 \leq s \leq t} (-X_s^{(+)}) \wedge 0 > y\}.$$  \hspace{1cm} (3.5)

It was shown in Pistorius (2004) that,

$$E[e^{-\delta \tau^{(+)}_y}] = \frac{1}{Z^{(y)}},$$  \hspace{1cm} (3.6)

where
where $Z^\delta(y) = 1 + \delta \int_0^y W^{(\delta)}(z)dz$ with $W^{(\delta)}(z)$ being the $\delta$-scale function, which is increasing and continuously differentiable with Laplace transform

$$
\int_0^\infty e^{-\delta z} W^{(\delta)}(z)dz = \frac{1}{\psi(\delta) - r}, \delta > \psi^{-1}(r),
$$

(3.7)

where $\psi^{-1}(r)$ is the inverse function of the Laplace exponent $\psi(\cdot)$. Furthermore, it holds that

$$
(Z^\delta(0))^\prime = \delta W^{(\delta)}(0) = 0.
$$

(3.8)

We can also conclude that the entrance times of the reflected processes $Y$ and $Y^{(+)}$ into $(y, +\infty)$ satisfy the following inequality

$$
\tau_y \leq \tau^{(+)}_y,
$$

(3.9)

that is because

$$
Y_i - Y^{(+)}_i = \sup_{0 \leq s \leq r} (X^{(+)}_s + X^{(-)}_s) \vee 0 - X^{(-)}_i - \sup_{0 \leq s \leq r} X^{(+)}_s \vee 0
$$

$$
= \sup_{0 \leq s \leq r} (X^{(+)}_s + X^{(-)}_s) \vee 0 + \sup_{0 \leq s \leq r} (-X^{(-)}_s) \vee 0 - \sup_{0 \leq s \leq r} X^{(+)}_s \vee 0 \geq 0.
$$

(3.10)

Now we present the smooth pasting property in the following Lemma.

**Lemma 3.3** Let $b > 0, \delta > 0$. If $\eta(0) < \infty, \int_0^\infty \eta(x)e^{-\beta x}dx < \infty, \quad \text{and} \quad \int_0^\infty x^{j-1}e^{-\gamma x}\eta(x)dx < \infty, i = 1, \ldots, m, j = 1, \ldots, n_i$, then $\Phi'(0^+) = 0$.

**Proof.** For $0 < y < b$, we have

$$
\Phi(0) = E[e^{-\delta Y_i \eta(Y_i \cdot b)}] = E[e^{-\delta Y_i} E_{Y_i}[e^{-\delta Y_i \eta(Y_i \cdot b)}]]
$$

$$
\geq E[e^{-\delta Y_i} 1_{\{Y_i = y\}} E_y[e^{-\delta Y_i \eta(Y_i \cdot b)}]] \geq E[e^{-\delta Y_i^{(+)}_i} 1_{\{Y^{(+)}_i = y\}}] \Phi(y) = E[e^{-\delta Y^{(+)}_i} 1_{\{Y^{(+)}_i = y\}}] \Phi(y),
$$

(3.11)

where the second inequality follows from $\tau_y \leq \tau^{(+)}_y$. So from (3.11) and (3.6), we have

$$
\Phi'(0^+) \leq 0.
$$

(3.12)
On the other hand,

\[
E[e^{-\delta t} \eta(Y_{t_0} - b)] = E[e^{-\delta t} E_{Y_{t_0}} [e^{-\delta \tau} \eta(Y_{\tau_0} - b)]]
\]

\[
= E[e^{-\delta \tau_0} 1_{(Y_{\tau_0} = y)} E_y [e^{-\delta \tau} \eta(Y_{\tau_0} - b)]] + E[e^{-\delta \tau_0} 1_{(Y_{\tau_0} > y)} E_y [e^{-\delta \tau} \eta(Y_{\tau_0} - b)]]
\]

\[
\leq E[e^{-\delta \tau_0} 1_{(Y_{\tau_0} = y)}] \Phi(y) + CP(Y_{\tau_0} > y),
\]

(3.13)

\[
C = \max \{\eta(0), \int_0^\infty \eta(x) e^{-\lambda x} dx, \int_0^\infty \frac{x^{j-1} \lambda^j e^{-\lambda x}}{(j-1)!} \eta(x) dx < \infty, i = 1, \ldots, m, j = 1, \ldots, n_i\},
\]

where the last inequality follows from Asmussen et al. (2004). Therefore

\[
\Phi'(0+) = \lim_{x \downarrow 0} \frac{\Phi(x) - \Phi(0)}{x} \geq \lim_{x \downarrow 0} \frac{\Phi(x)(1 - P(Y_{\tau_0} = x)) - C(1 - P(Y_{\tau_0} = x))}{x}.
\]

(3.14)

Because

\[
0 \leq \lim_{x \downarrow 0} \frac{1 - P(Y_{\tau_0} = x)}{x} \leq \lim_{x \downarrow 0} \frac{1 - E[e^{-\lambda (1-P) \tau_0^{+}}]}{x} = 0.
\]

It follows that \( \Phi'(0+) \geq 0 \), which concludes the proof.

The next result gives the infinitesimal generator of the process \( Y = M - X \).

**Lemma 3.4** Let \( X \) be the two-sided jump-diffusion model given by (2.1). Then the infinite-simal generator of the reflected Levy process \( Y = M - X \) is given by

\[
Lu(x) = -cu'(x) + \frac{1}{2} \sigma^2 u''(x) + \lambda \int_{-\infty}^{\infty} [u((x - z) \wedge 0) - u(x)] f(z) dz,
\]

(3.15)

with the domain of definition

\[
D(A) = \{u \in C^2 : u'(0) = 0\}.
\]

**Proof.** Since the proof of Lemma 3.4 is similar to the proof of Proposition 3.1 in Bo et al. (2012), we omit the proof here.
4. Laplace Transforms

In this section, we will derive the explicit expression for the joint Laplace transform of the ruin time and the deficit at ruin time under the two-sided jump-diffusion model.

**Theorem 4.1** For $\delta > 0, \eta \leq 0$, we have

$$E[e^{-\delta r_t + \eta (Y_t - b)} | Y_0 = x] = \sum_{i=1}^{n+4} c_i e^{-r_i (x-b)}, \quad 0 < x < b,$$

(4.1)

where $r_1, \cdots, r_{n+4}$ are $n+4$ distinct roots of the Eq. $g(x) = \delta$ on the whole complex plane and $c_i$'s satisfy the following linear system

$$\begin{cases}
\sum_{i=1}^{n+4} c_i = 1, \\
\sum_{i=1}^{n+4} c_i r_i e^{rb} = 0, \\
\sum_{i=1}^{n+4} c_i r_i e^{rb} = 0, \\
\sum_{i=1}^{n+4} c_i (\beta - \eta) = 1, \\
\sum_{i=1}^{n+4} c_i (v_i + r_i)^j = 1, l = 1, \cdots, m, j = 1, \cdots, n_i.
\end{cases}$$

(4.2)

**Proof.** The proof is similar to that of Theorem 3.1 in Kou and Wang (2003). For any fixed $x > 0$, define the function $u(x)$ to be

$$u(x) = \begin{cases}
\sum_{i=1}^{n+4} c_i e^{-r_i (x-b)}, & 0 < x < b, \\
e^{\eta (x-b)}, & x \geq b.
\end{cases}$$

Clearly, $u(x)$ is continuous and bounded for all $x \in (0, \infty)$. So, there exists a constant $M \geq 1$, such that $|u(x)| \leq M$ for all $x \in (0, \infty)$. For $0 < x < b$, $Lu(x) - \delta u(x) = \sum_{i=1}^{n+4} g(r_i) e^{-r_i (x-b)} - \lambda p_i \sum_{i=1}^{n+4} c_i r_i e^{-\alpha r_i + rb} + \lambda \sum_{k=1}^{n} c_k \sum_{j=1}^{m} q_{ij}$
\[
\sum_{j=1}^{n} v_j (b - x)^{j-1} e^{v_j (x-b)} (j-1)! \left( \frac{1}{v_j - \eta} \right) = \frac{1}{(v_j - \eta - \frac{c_i \beta}{\beta + r_j})} + \lambda q_i \left( \frac{\beta}{\beta - \eta} - \sum_{i=1}^{n+4} \frac{c_i \beta}{\beta + r_j} \right) e^{v_j (x-b)},
\]

from (4.2) and \( g(r_i) = 0 \), we conclude

\[
Lu(x) - \delta u(x) = 0, \quad 0 < x < b.
\]  

(4.4)

Because the function \( u(x) \) is not differentiable at \( x = b \), we cannot apply \( \hat{I}to \) formula directly to the process \( \{e^{-\delta t}u(Y_t); t \geq 0\} \). But there exists a sequence of functions \( \{u_m(x); m = 1, 2, \ldots\} \) such that: (i) \( u_m(x) \) is smooth everywhere, and in particular it belongs to \( C^2 \); (ii) \( u_m(x) = u(x) \) for all \( 0 \leq x \leq b \); (iii) \( u_m(x) = u(x) \) for all \( x \geq b + 1/m \); (iv) \( 0 \leq u_m(x) \leq 2 \) for \( x \in (b, b + 1/m) \).

Obviously, \(|u_m(x)| \leq M + 1\) for all \( x, m \) and \( u_m(x) \to x \) for all \( x \).

After some calculations we obtain for \( 0 < x < b \),

\[
Lu_m(x) = \frac{1}{2} \sigma^2 u_m''(x) - cu_m'(x) + \lambda \int_{-\infty}^{x} [u_m((x-y) \vee 0) - u_m(x)] f(y) dy
\]

\[
= \frac{1}{2} \sigma^2 u_m''(x) - cu_m'(x) - \lambda \int_{-\infty}^{x-b-1/m} u_m(x-y) f(y) dy
\]

\[
+ \lambda \int_{x-b}^{x-b-1/m} u_m(x-y) f(y) dy + \lambda \int_{x-b}^{x} u_m((x-y) \vee 0) f(y) dy
\]

\[
= \delta u(x) + \lambda \int_{x-b-1/m}^{x-b} \left[ u_m(x-y) - u(x-y) \right] f(y) dy,
\]

(4.5)

where the last equality follows from (4.4). Since \(|u_m - u| \leq 2\) by construction, we have

\[
| -\delta u_m(x) + Lu_m(x) | \leq \lambda \int_{x-b-1/m}^{x-b} \left| u_m(x-y) - u(x+y) \right| (|q_i| |\beta + \sum_{i=1}^{n} \sum_{j=1}^{q_{ij}} v_{ij}|) dy
\]

\[
\leq \frac{(|q_i| |\beta + \sum_{i=1}^{n} \sum_{j=1}^{q_{ij}} v_{ij}|)}{m} \to 0, \quad 0 < x < b,
\]

(4.6)

uniformly in \( x \) as \( m \to \infty \). Applying \( \hat{I}to \) formula to the process \( \{e^{-\delta t}u_m(Y_t); t \geq 0\} \), we obtain that the process
\[ M_i^{(m)} := e^{-\delta(t \wedge \tau_b)} u_m(Y_{t \wedge \tau_b}) - \int_0^{t \wedge \tau_b} e^{-\delta s} (-\delta u_m(Y_s) + Lu_m(Y_s))ds, \quad t \geq 0, \]

is a local martingale starting from \( M_0^{(m)} = u_m(x) = u(x) \). And from (4.6), we deduce that

\[ |M_t^m| \leq M + 1 + \frac{t \sum_{i=1}^{m} \sum_{j=1}^{m} |q_{ij}|}{m}, \quad t \geq 0, \]

so from the dominated convergence theorem that \( \{M_i^{(m)}; t \geq 0\} \) is actually a martingale. Then

\[ E[M_i^m] = E[e^{-\delta(t \wedge \tau_b)} u_m(Y_{t \wedge \tau_b}) - \int_0^{t \wedge \tau_b} e^{-\delta s} (-\delta u_m(Y_s) + Lu_m(Y_s))ds] = u(x), \]

for all \( t \geq 0 \). Letting \( m \to \infty \), it follows from the dominated convergence theorem that

\[ \lim_{m \to \infty} E[e^{-\delta(t \wedge \tau_b)} u_m(Y_{t \wedge \tau_b})] = E[e^{-\delta(t \wedge \tau_b)} u(Y_{t \wedge \tau_b})], \]

and from (4.6), we conclude

\[ \lim_{m \to \infty} E\left[ \int_0^{t \wedge \tau_b} e^{-\delta s} (-\delta u_m(Y_s) + Lu_m(Y_s))ds \right] = 0. \]

Hence, for any \( t \geq 0 \),

\[ u(x) = E[e^{-\delta(t \wedge \tau_b)} u(Y_{t \wedge \tau_b})] = E[e^{-\delta(t \wedge \tau_b)} u(Y_{t \wedge \tau_b})1_{\{\tau_b < \infty\}}] + E[e^{-\delta \tau_b} u(Y_{\tau_b})1_{\{\tau_b < \infty\}}]. \quad (4.7) \]

Then by letting \( t \to \infty \), we have

\[ u(x) = E[e^{-\delta \tau_b} u(Y_{\tau_b})1_{\{\tau_b < \infty\}}] = E[e^{-\delta \tau_b + \delta(Y_{\tau_b} - b)}1_{\{\tau_b < \infty\}}], \quad (4.8) \]

as \( u(Y_{\tau_b}) = 1 \) on the set \( \{\tau_b < \infty\} \). This completes the proof.

From Theorem 4.1, we can give the explicit formula for the joint Laplace transform of ruin time and the deficit in the next result.

**Corollary 4.1** For \( \delta > 0 \), we have
where $r_1, \cdots, r_{n+4}$ are $n+4$ distinct roots on the whole complex plane of the Eq. $g(x) = \delta$, and $c_i$'s are determined by (4.2).

The next result presents the Laplace transform of ruin time, which is a special case of the joint Laplace transform by letting $\eta = 0$ in $\Psi(u)$.

**Corollary 4.2** For $\delta > 0$, we have

$$
E[e^{-\delta\tau} \mid U(0) = u] = \sum_{i=1}^{n+4} c_i e^{ir_u}, \quad 0 < u < b,
$$

(4.9)

where $r_1, \cdots, r_{n+4}$ are $n+4$ distinct roots of the Eq. $g(x) = \delta$ on the whole complex plane and $c_i$'s satisfy the following linear system

$$
\begin{align*}
\sum_{i=1}^{n+4} c_i &= 1, \\
\sum_{i=1}^{n+4} c_i r_i e^{ir_b} &= 0, \\
\sum_{i=1}^{n+4} c_i r_i e^{ir_b} &= 0, \\
\sum_{i=1}^{n+4} c_i \alpha - r_i &= 0, \\
\sum_{i=1}^{n+4} c_i \beta - r_i &= 1, \\
\sum_{i=1}^{n+4} c_i (v_i + r_i)^j &= 1, l = 1, \cdots, m, j = 1, \cdots, n.
\end{align*}
$$

(4.11)

From Corollary 4.2, we can easily obtain the infinite-time ruin probability.

**Corollary 4.3** For $0 \leq u \leq b$, we have

$$
P(\hat{\tau}_u < \infty) = 1.
$$

(4.12)

**Proof.** By Lemma 2.1 and the condition $ct + E[S(t)] > 0$, we have $r_{n+3} \to 0$, when $\delta \to 0$. So from (4.11), we have $d_{n+3} = 1$, and $d_i = 0, i = 1, 2, \cdots, n+2, n+4$. Therefore,
\[ P(\hat{\tau}_u < \infty) = \lim_{\delta \to 0} E[e^{-\delta \hat{\tau}_u} \mid U(0) = u] = 1. \]

The following result provides an expression for the joint Laplace transform of ruin time and the deficit at ruin time without dividend.

**Corollary 4.4** For \( \delta > 0, \eta < 0 \), the joint Laplace transform of ruin time and the deficit at ruin time without dividend is given by

\[
E[e^{-\delta \hat{\tau}_u - \eta (U_0)} \mid U(0) = u] = \sum_{i=1}^{n+2} l_i e^{\gamma_i u}, \quad u > 0, \tag{4.13}
\]

where \( r_1, \ldots, r_{n+2} \) are the roots on the left complex plane of the Eq. \( g(x) = \delta \), and \( l_i \)'s satisfy

\[
\begin{cases}
\sum_{i=1}^{n+2} l_i = 1, \\
\sum_{i=1}^{n+2} \frac{l_i (\beta - \eta)}{\beta + r_i} = 1, \\
\sum_{i=1}^{n+2} \frac{l_i (v_i - \eta)^j}{(v_i + r_i)^j} = 1, & l = 1, \ldots, m, \quad j = 1, \ldots, n_i.
\end{cases}
\tag{4.14}
\]

**Proof.** The equality (4.13) can be obtained from (4.1) and (4.2) by letting \( b \to +\infty \).

### 5. Numerical Calculations and Conclusion

In this section, we will use the Gaver-Stehfest algorithm (see Kou and Wang (2003)) inverting the Laplace transform to make some numerical analysis. In our calculations, the density \( f_2(y) \) is chosen as in Example 4.1. Let \( \nu_0 = 80, \alpha = \beta = 160, \quad p_0 = 0.5, \quad c = 0.4, \sigma = 0.2, \quad \lambda_1 = 12, \quad \lambda_{12} = 8, \quad \lambda_2 = 30 \). From Figs. 1, 2, we can easily see the ruin probability is a decreasing function of initial surplus \( u \), and for fixed \( u \), the ruin probability increases with \( b \) decreasing.
This article considers a two-sided jump-diffusion model incorporating two dependent classes of jumps under a constant dividend barrier. We give the smooth pasting condition for the expected discounted penalty function. Based on this condition, we obtain the explicit expression for the Laplace transform of the ruin time and the deficit under the two-sided jump-diffusion model. In addition, using the Gaver-Stehfest inversion algorithm, we give some numerical analysis.

References


